Further mathematics
Higher level
Paper 1

Thursday 17 May 2018 (afternoon)

2 hours 30 minutes

Instructions to candidates

- Do not open this examination paper until instructed to do so.
- Answer all questions.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A graphic display calculator is required for this paper.
- A clean copy of the **mathematics HL and further mathematics HL formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[150 marks]**.

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Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. In particular, solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 9]

(a) Use the Euclidean algorithm to find the greatest common divisor of 74 and 383. [4]

(b) Hence find integers s and t such that $74s + 383t = 1$. [5]

2. [Maximum mark: 6]

Let $\boldsymbol{A}^2 = 2\boldsymbol{A} + \boldsymbol{I}$ where $\boldsymbol{A}$ is a 2×2 matrix.

(a) Show that $\boldsymbol{A}^4 = 12\boldsymbol{A} + 5\boldsymbol{I}$. [3]

Let $\boldsymbol{B} = \begin{bmatrix} 4 & 2 \\ 1 & -3 \end{bmatrix}$.

(b) Given that $\boldsymbol{B}^2 - \boldsymbol{B} - 4\boldsymbol{I} = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$, find the value of k. [3]

3. [Maximum mark: 7]

(a) A number written in base 5 is 4303. Find this as a number written in base 10. [2]

(b) 1000 is a number written in base 10. Find this as a number written in base 7. [5]

4. [Maximum mark: 12]

The transformations T_1, T_2, T_3, T_4, in the plane are defined as follows:

T_1: A rotation of $360°$ about the origin
T_2: An anticlockwise rotation of $270°$ about the origin
T_3: A rotation of $180°$ about the origin
T_4: An anticlockwise rotation of $90°$ about the origin.

(a) Copy and complete the following Cayley table for the transformations of T_1, T_2, T_3, T_4, under the operation of composition of transformations.

	T_1	T_2	T_3	T_4
T_1	T_1	T_2	T_3	T_4
T_2	T_2			
T_3	T_3			
T_4	T_4			

[2]

(b) (i) Show that T_1, T_2, T_3, T_4 under the operation of composition of transformations form a group. Associativity may be assumed.

(ii) Show that this group is cyclic. [4]

The transformation T_5 is defined as a reflection in the x-axis.

(c) Write down the 2×2 matrices representing T_3, T_4 and T_5. [3]

(d) The transformation T is defined as the composition of T_3 followed by T_5 followed by T_4.

(i) Find the 2×2 matrix representing T.

(ii) Give a geometric description of the transformation T. [3]

5. [Maximum mark: 7]

Use the integral test to determine whether or not $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ converges. [7]

6. [Maximum mark: 9]

(a) Consider the integers between 1 and 20 inclusive.
Let $A = \{$multiples of 2$\}$, $B = \{$multiples of 3$\}$, $C = \{$multiples of 4$\}$.
Find the elements in each of the following sets,

(i) $A \cap (B \cup C)$;

(ii) $A \setminus (B \setminus C)$. [5]

(b) Let $M = \{x : x$ is an integer multiple of 10$\}$ and let $N = \{x : x$ is an integer multiple of 5$\}$
Prove that M is a proper subset of N. [4]

7. [Maximum mark: 9]

A sample of size 100 is taken from a normal population with unknown mean μ and known variance 36.

(a) An investigator wishes to test the hypotheses $H_0 : \mu = 65$, $H_1 : \mu > 65$.

He decides on the following acceptance criteria:
Accept H_0 if the sample mean $\bar{x} \le 66.5$
Accept H_1 if $\bar{x} > 66.5$

Find the probability of a Type I error. [3]

(b) Another investigator decides to use the same data to test the hypotheses
$H_0 : \mu = 65$, $H_1 : \mu = 67.9$.

(i) She decides to use the same acceptance criteria as the previous investigator.
Find the probability of a Type II error.

(ii) Find the critical value for $\bar{x}$ if she wants the probabilities of a Type I error and a Type II error to be equal. [6]

8. [Maximum mark: 13]

Consider the simultaneous linear equations

$$x + z = -1$$
$$3x + y + 2z = 1$$
$$2x + ay - z = b$$

where a and b are constants.

(a) Using row reduction, find the solutions in terms of a and b when $a \neq 3$. [8]

(b) Explain why the equations have no unique solution when $a = 3$. [1]

(c) Find all the solutions to the equations when $a = 3$, $b = 10$ in the form $\boldsymbol{r} = \boldsymbol{s} + \lambda\boldsymbol{t}$. [4]

9. [Maximum mark: 13]

(a) Given that A is the interval $\{x : 0 \le x \le 3\}$ and B is the interval $\{y : 0 \le y \le 4\}$ then describe $A \times B$ in geometric form. [3]

(b) Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ be defined by $f(x, y) = (x + 3y, 2x - y)$.

(i) Show that the function f is a bijection.

(ii) Hence find the inverse function f^{-1}. [10]

10. [Maximum mark: 12]

(a) By considering the images of the points (1, 0) and (0, 1),

(i) determine the 2×2 matrix $\boldsymbol{P}$ which represents a reflection in the line $y = (\tan\theta)x$;

(ii) determine the 2×2 matrix $\boldsymbol{Q}$ which represents an anticlockwise rotation of θ about the origin. [5]

(b) Describe the transformation represented by the matrix $\boldsymbol{PQ}$. [5]

(c) A matrix M is said to be orthogonal if $\boldsymbol{M}^T\boldsymbol{M} = \boldsymbol{I}$ where $\boldsymbol{I}$ is the identity. Show that $\boldsymbol{Q}$ is orthogonal. [2]

11. [Maximum mark: 12]

Given that y is a function of x, the function z is given by $z = \frac{y-x}{y+x}$, where $x \in \mathbb{R}, x \neq 3$, $y + x \neq 0$.

(a) Show that $\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{2}{(y+x)^2}\left(x\frac{\mathrm{d}y}{\mathrm{d}x} - y\right)$. [3]

(b) Show that the differential equation $f(x)\left(x\frac{\mathrm{d}y}{\mathrm{d}x} - y\right) = y^2 - x^2$ can be written as $f(x)\frac{\mathrm{d}z}{\mathrm{d}x} = 2z$. [2]

(c) Hence show that the solution to the differential equation $(x-3)\left(x\frac{\mathrm{d}y}{\mathrm{d}x} - y\right) = y^2 - x^2$ given that $x = 4$ when $y = 5$ is $\frac{y-x}{y+x} = \left(\frac{x-3}{3}\right)^2$. [7]

12. [Maximum mark: 15]

(a) Solve the recurrence relation $u_n = 4u_{n-1} - 4u_{n-2}$ given that $u_0 = u_1 = 1$. [6]

Consider v_n which satisfies the recurrence relation $2v_n = 7v_{n-1} - 3v_{n-2}$ subject to the initial conditions $v_0 = v_1 = 1$.

(b) Prove by using strong induction that $v_n = \frac{4}{5}\left(\frac{1}{2}\right)^n + \frac{1}{5}(3)^n$ for $n \in \mathbb{N}$. [9]

13. [Maximum mark: 9]

Consider the matrix $\boldsymbol{M} = \begin{bmatrix} 2 & -4 \\ -1 & -1 \end{bmatrix}$.

(a) Show that the linear transformation represented by $\boldsymbol{M}$ transforms any point on the line $y = x$ to a point on the same line. [2]

(b) Explain what happens to points on the line $4y + x = 0$ when they are transformed by $\boldsymbol{M}$. [3]

(c) State the two eigenvalues of $\boldsymbol{M}$. [2]

(d) State two eigenvectors of $\boldsymbol{M}$ which correspond to the two eigenvalues. [2]

14. [Maximum mark: 8]

At an early stage in analysing the marks scored by candidates in an examination paper, the examining board takes a random sample of 250 candidates and finds that the marks, x, of these candidates give $\sum x = 10985$ and $\sum x^2 = 598736$.

(a) Calculate a 90% confidence interval for the population mean mark μ for this paper. [4]

(b) The null hypothesis $\mu = 46.5$ is tested against the alternative hypothesis $\mu < 46.5$ at the $\lambda\%$ significance level. Determine the set of values of λ for which the null hypothesis is rejected in favour of the alternative hypothesis. [4]

15. [Maximum mark: 9]

Given that the tangents at the points P and Q on the parabola $y^2 = 4ax$ are perpendicular, find the locus of the midpoint of PQ. [9]